Level statistics in arithmetical and pseudo-arithmetical chaos

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We resolve a long-standing riddle in quantum chaos, posed by certain fully chaotic billiards with constant negative curvature whose periodic orbits are highly degenerate in length. Depending on the boundary conditions for the quantum wave functions, the energy spectra either have uncorrelated levels usually associated with classical integrability or conform to the "universal" Wigner-Dyson type although the classical dynamics in both cases is the same. The resolution turns out surprisingly simple. The Maslov indices of orbits within multiplets of degenerate length either yield equal phases for the respective Feynman amplitudes (and thus Poissonian level statistics) or give rise to amplitudes with uncorrelated phases (leading to Wigner-Dyson level correlations). The recent semiclassical explanation of spectral universality in quantum chaos is thus extended to the latter case of "pseudo-arithmetical" chaos.

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Introduction: After more than two decades of investigations, the famous BGS conjecture [1] has recently found a semiclassical explanation[2–6]: The quantum level spectra of classically chaotic systems display fluctuations conforming to the random-matrix-theory (RMT) predictions for the Wigner-Dyson universality classes.

However, there is the notable exception of "arithmetical" systems which are fully chaotic classically but display quantum spectral statistics close to Poissonian, a behavior usually associated with integrable classical motion [7–12]. Quantum mechanically these exceptional dynamics exhibit an infinite number of the so called Hecke operators commuting with the Hamiltonian. Therefore the energy spectrum falls into practically independent multiplets such that nearby levels bear no correlation. On the classical side the periodic-orbit action spectra of such systems are distinguished by a degeneracy exponentially growing with the orbit period.

On the other hand, by merely changing the boundary conditions for the quantum wave functions of some such exceptional arithmetical systems one can retrieve universal spectral fluctuations à la Wigner and Dyson while not at all changing the classical dynamics. It is customary to speak of pseudo-arithmetical systems then [13–15].

The strikingly different quantum behavior of arithmetical and pseudo-arithmetical systems might raise doubts about the validity of the recent semiclassical explanation of universal quantum spectral fluctuations under conditions of classical chaos; after all, the classical dynamics are identical for the systems under discussion, and so appear, on first sight, the Gutzwiller type semiclassical periodic-orbit expansions. Various suggestions were ventured for the effect of the boundary conditions, among them a distinction between the orbit classes contributing to the Selberg trace formula applicable (and exact) in the arithmetical case, and the Gutzwiller formula applicable in the pseudo-arithmetical case.

We show here that the explanation is much simpler and lies in special properties of periodic orbits. Due to these peculiarities all equal-length orbits (save for a negligible fraction) of an arithmetical system contribute Feynman amplitudes with the same Maslov phase; their constructive interference makes for nonuniversal spectral statistics. In the pseudo-arithmetical case these phases vary randomly within a degenerate-action multiplet such that destructive interference makes the high action degeneracy ineffective. The difference between the two cases is most easily revealed for the diagonal approximation to the spectral form factor, and therefore that approximation will play a central role here; off-diagonal corrections will be discussed briefly in the end.

The billiard $T^*(2,3,8)$: We shall not deal with the alternative arithmetical/pseudo-arithmetical in full generality but prefer to work with a representative example, the so called triangular billiard $T^*(2,3,8)$. That system was first considered in the studies of the free motion on the surface of constant negative curvature tesselated by regular octagons[16]. Desymmetrization of the regular octagon necessary to get rid of the rotational and reflection symmetry leads to a triangular fundamental domain with the angles $\pi/2, \pi/3, \pi/8$. Depicted inside the Poincaré disk $|z| \le 1$ in the complex plane, the triangle has its hypotenuse (N) and the longer leg (L) directed along two diameters whereas the shorter leg (M) looks like an arc (Fig. 1). A classical periodic orbit folded into the triangle looks like a sequence of arcs mirror-reflected from the sides of the triangle. The classical motion is completely chaotic, all periodic orbits having the same Lyapunov constant. The multiplicities in the length spectrum of periodic orbits grow exponentially with the length, like $\exp(l_\gamma/2)$, where l_γ is made dimensionless by referral to a scale fixed by setting the curvature to -1; since action is proportional to the orbit length the action spectrum also has an exponentially growing degeneracy.

The quantum energy levels for $T^*(2,3,8)$ are found as eigenvalues of the Laplace-Beltrami operator. The

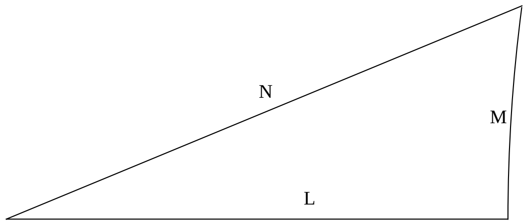

FIG. 1: Fundamental domain of $T^*(2,3,8)$

boundary conditions can be either Dirichlet or Neumann. There are $2^3 = 8$ quantum mechanical problems, all related to the same classical system. In problems stemming from desymmetrization of the regular octagon, the boundary conditions on the triangle sides are chosen to obtain the spectrum for a particular irreducible representation. The boundary conditions on the hypotenuse N and the shorter leg M must then be the same, i.e., both Dirichlet or both Neumann; four such possibilities exist all of which lead to arithmetical systems with near-Poissonian spectral statistics. The remaining four choices where N and M host different boundary conditions, lead to pseudo-arithmetical systems with the Wigner-Dyson statistics of the orthogonal universality class.

Form factor and diagonal approximation: The spectral form factor following from Gutzwiller's trace formula is a double sum over periodic orbits,

$$K(\tau) \sim \left\langle \sum_{\gamma,\gamma'} A_\gamma A_{\gamma'} \exp\left[i\frac{S_\gamma - S_{\gamma'}}{\hbar} - i\frac{(\mu_\gamma - \mu_{\gamma'})\pi}{2}\right] \times \delta\left(\tau T_H - \frac{T_\gamma + T_{\gamma'}}{2}\right)\right\rangle, \tag{1}$$

where $S_\gamma, T_\gamma, \mu_\gamma, A_\gamma = A_\gamma^*$ are action, period, Maslov index, and stability coefficient of the orbit γ; the Heisenberg time $T_H = \frac{2\pi\hbar}{\Delta}$, with Δ the mean level spacing, is used as a unit of time such that τ becomes a dimensionless time; the angular brackets $\langle\ldots\rangle$ denote averages over the energy shell and over a small τ interval.

Of special interest are the pairs of orbits obviously immune against destructive interference of their contributions, namely with the same action and Maslov index. For generic chaotic dynamics these are the trivial pairs $\gamma' = \gamma$ and, if time reversal is allowed, the pairs of mutually time reversed orbits, $\gamma' = \gamma^{\mathrm{TR}}$. Discarding all other pairs one gets Berry's diagonal approximation [2],

$$K_{\mathrm{diag}}(\tau) \sim \sum_\gamma A_\gamma^2 \delta(\tau T_H - T_\gamma) = g\tau\,; \tag{2}$$

here g is the average multiplicity of the action spectrum which is 1 in the absence of time reversal (unitary universality class) or 2 in the presence of time reversal (orthogonal class); the result yields the first-order term of a power series in τ, in agreement with RMT [4].

Turning to arithmetical chaos we can carry over the foregoing reasoning, except that the multiplicity is now exponentially large, $g \propto \mathrm{e}^{l/2}$. However, orbits with the same length bear no geometric similarity, and it is not at all obvious that their Maslov phases are the same.

As will be shown, the Maslov phases do indeed coincide for all orbits in a length multiplet, a negligible fraction apart, in the arithmetical case. Consequently, the form factor exhibits almost instant increase at $\tau \geq 0$, similar to the jump of the integrable case [6], $K(0) = 0$ and $K(\tau) = 1$ for $\tau > 0$.

In contrast, the Maslov index of pseudo-arithmetical systems will turn out to fluctuate randomly within each fixed-action multiplet. The usual diagonal approximation with $g = 2$ then holds since in each multiplet only the pairs $\gamma' = \gamma$ and $\gamma' = \gamma^{\mathrm{TR}}$ escape destructive interference. The diagonal approximation thus suggests universal spectral fluctuations.

The Maslov index of an orbit in our billiard is determined only by the number N_D of reflections from the sides with the Dirichlet boundary condition: each such reflection changes the Maslov phase by π. Therefore the contribution of an equal-action pair γ, γ' is $A_\gamma^2 (-1)^{N_D - N_D'}$ where N_D, N_D' respectively refer to γ and γ'; in long orbits both N_D and N_D' are large pseudo-random integers. We shall demonstrate that in arithmetical systems equal-length orbits have, in their overwhelming majority, N_D of the same parity (even or odd) such that contributions of all pairs of them are positive and add up. On the contrary, pseudo-arithmetical systems have uncorrelated parities of N_D, N_D' and the equal-action contributions mutually cancel, apart from the standard pairs of the orthogonal universality class.

Numerical observations: The triangular billiard affords symbolic dynamics, and calculating its orbits involves generating all allowed sequences of symbols L, M, N each standing for the visit of the respective side. We denote by $\lambda_\gamma, \mu_\gamma, \nu_\gamma$ the number of symbols L, M, N in an orbit γ; the total number of symbols in γ is $n_\gamma = \lambda_\gamma + \mu_\gamma + \nu_\gamma$. Studying up to a million orbits we find:

— Orbits within a given length multiplet Λ almost always have n_γ with the same parity. We can therefore speak about Λ_g- and Λ_u-multiplets depending on parity of $n_\gamma, \gamma \in \Lambda$. Exceptions are extremely rare and in fact amount to a negligible fraction: E.g., among approximately 257000 orbits with the length $l_\gamma < 16$ grouped into more than 13000 length multiplets, only 4 multiplets are "*gu*-degenerate", i.e., contain orbits with both even and odd number of symbols; these offenders have lengths $l = 10.6999964,\ 12.2422622, 13.7571382,\ 15.2857092$.

— *All* orbits in a given Λ without *gu*-degeneracy have λ_γ of the same parity which automatically leads to definite parity of $\mu_\gamma + \nu_\gamma$. On the other hand, μ_γ and ν_γ separately have no definite parity within Λ.

These observations, in particular the rarity of multiplets with *gu*-degeneracy, suffice to explain the sensitivity

of the level statistics to the boundary conditions. We denote by Φ_L the phase jump on reflection from the side L, which is 0 for the Neumann and π for the Dirichlet condition on L; similarly, Φ_M, Φ_N denote the phase jumps on M and N. The contribution to the diagonal form factor of a pair of orbits $(\gamma\gamma')$ belonging to the same Λ is

$$\begin{aligned} K_{\gamma\gamma'} &= A_\Lambda^2 \mathrm{e}^{i\left(\lambda_\gamma-\lambda_{\gamma'}\right)\Phi_L+i\left(\mu_\gamma-\mu_{\gamma'}\right)\Phi_M+i\left(\nu_\gamma-\nu_{\gamma'}\right)\Phi_N} \\ &= A_\Lambda^2 \mathrm{e}^{i\left(\mu_\gamma-\mu_{\gamma'}\right)\Phi_M+i\left(\nu_\gamma-\nu_{\gamma'}\right)\Phi_N}; \end{aligned}$$

the phase proportional to Φ_L disappears since, in the absence of the gu-degeneracy, $\lambda_\gamma - \lambda_{\gamma'}$ is even. If the boundary conditions on M and N are different (the pseudo-arithmetical case) one of Φ_M, Φ_N is zero and another one π. The contribution $K_{\gamma\gamma'}$ can then be of any sign and, summed over all pairs of a large multiplet, except the standard $\gamma' = \gamma, \gamma^{\mathrm{TR}}$, averages to zero. The form factor will be the Wigner-Dyson one for the orthogonal universality class, $g = 2$.

On the other hand, in arithmetical systems with the same boundary conditions on the sides M, N we have $\Phi_M = \Phi_N \equiv \Phi_{MN}$, and the contribution of every pair within Λ will be positive since $\mu_\gamma + \nu_\gamma - \mu_{\gamma'} - \nu_{\gamma'}$ is even,

$$K_{\gamma\gamma'} = A_\Lambda^2 \mathrm{e}^{i\left(\mu_\gamma+\nu_\gamma-\mu_{\gamma'}-\nu_{\gamma'}\right)\Phi_{MN}} = A_\Lambda^2.$$

Eq.(2) then applies with the abnormally high value of g bringing about the nearly vertical rise of $K(\tau)$ at $\tau \geq 0$.

Analytic reasoning: Using group-theoretical properties [15] of $T^*(2,3,8)$ we have substantiated these results analytically. The symbols L, M, N can be associated with transformations mapping the interior of the Poincaré disk $|z| \leq 1$ onto itself. Each elementary operation involves complex conjugation $K : z \to z^*$ followed by a Möbius transformation $z \to (az+b)/(b^*z+a^*)$; the three respective matrices $\left(\begin{smallmatrix} a & b \\ b^* & a^* \end{smallmatrix}\right)$ can be chosen as

$$\begin{aligned} \rho_L &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ \rho_M &= \begin{pmatrix} i\left(1+\frac{\sqrt{2}}{2}\right)\gamma & -i\frac{\sqrt{2}}{2}\beta \\ i\frac{\sqrt{2}}{2}\beta & -i\left(1+\frac{\sqrt{2}}{2}\right)\gamma \end{pmatrix}, \\ \rho_N &= \begin{pmatrix} \left(\frac{1+\sqrt{2}}{2}+\frac{i}{2}\right)\gamma & 0 \\ 0 & \left(\frac{1+\sqrt{2}}{2}-\frac{i}{2}\right)\gamma \end{pmatrix} \end{aligned} \tag{3}$$

where α, β, γ denote the quartic irrationalities

$$\alpha = \sqrt{\sqrt{2}-1}, \quad \beta = \sqrt{\sqrt{2}}, \quad \gamma = \sqrt{2-\sqrt{2}}\,. \tag{4}$$

Let us write the code of the orbit γ starting with an arbitrary symbol and multiply the associated elementary operators. The product is either a pure Möbius transformation with a certain matrix ρ_γ, if the number of symbols n_γ is even, or K followed by ρ_γ (odd n_γ). The cumulative transformation leaves invariant a circle in the complex plane (an invariant geodesic of the transformation) crossing the fundamental domain; its part inside the domain is the piece of γ between the two bounces against the sides given by the first and last symbol of the code. Cyclically shifting the code by one symbol one analogously gets the next orbit piece, and so forth [15]. The matrix ρ_γ yields the orbit length as

$$\begin{aligned} 2\cosh\frac{l_\gamma}{2} &= \mathrm{Tr}\rho_\gamma, \quad n_\gamma \text{ even}, \\ 2\sinh\frac{l_\gamma}{2} &= \mathrm{Atr}\,\rho_\gamma, \quad n_\gamma \text{ odd}, \end{aligned} \tag{5}$$

where $\mathrm{Atr}\,\rho$ is the sum of the off-diagonal elements of ρ.

It can be shown by induction that the matrices ρ_γ can be of two arithmetical types ([15], Appendix B),

$$\begin{aligned} \rho^{(1)} &= \begin{pmatrix} \frac{1}{2}(u_{1,R}+iu_{1,I}) & \frac{1}{2}(v_{1,R}+iv_{1,I})\,\alpha \\ \frac{1}{2}(v_{1,R}-iv_{1,I})\,\alpha & \frac{1}{2}(u_{1,R}-iu_{1,I}) \end{pmatrix}, \\ \rho^{(2)} &= \begin{pmatrix} \frac{1}{2}(u_{2,R}+iu_{2,I})\,\gamma & \frac{1}{2}(v_{2,R}+iv_{2,I})\,\beta \\ \frac{1}{2}(v_{2,R}-iv_{2,I})\,\beta & \frac{1}{2}(u_{2,R}-iu_{2,I})\,\gamma \end{pmatrix}. \end{aligned} \tag{6}$$

Here $u_{k,R}, v_{k,R}, u_{k,I}, v_{k,I}$ are algebraic integers,

$$\begin{aligned} u_{k,R} &= m_{k,R} + n_{k,R}\sqrt{2}, \\ v_{k,R} &= p_{k,R} + q_{k,R}\sqrt{2}, \quad k = 1,2, \end{aligned} \tag{7}$$

with integers $m_{k,R}, n_{k,R}, p_{k,R}, q_{k,R}$; the imaginary parts have the same appearance. Appending a symbol L (pure complex conjugation) to the code of the orbit doesn't change the type of ρ_γ whereas appending M or N toggles the type, $\rho^{(1)} \leftrightarrow \rho^{(2)}$. The matrices ρ_M and ρ_N belong to the type $\rho^{(2)}$; therefore ρ_γ belongs to the type $\rho^{(1)}$ if the sum $\mu_\gamma + \nu_\gamma$ of the number of symbols M, N in the code of γ is even, and ρ_2 if $\mu_\gamma + \nu_\gamma$ is odd.

Eqs. (5,6) entail *four* types of orbit lengths: Orbits with even numbers of symbols n_γ have lengths

$$\text{(a)}\quad 2\cosh\frac{l_g}{2} = u_{1,R}\,, \qquad \text{(b)}\quad 2\cosh\frac{l_g}{2} = \gamma\, u_{2,R}\,, \tag{8}$$

while orbits with odd n_γ have lengths

$$\text{(c)}\quad 2\sinh\frac{l_u}{2} = \alpha\, v_{1,R}\,, \qquad \text{(d)}\quad 2\sinh\frac{l_u}{2} = \beta\, v_{2,R}\,. \tag{9}$$

The integer components of u, v in these equations are restricted [15] by the inequalities

$$\begin{aligned} &\text{(a)}\quad |m - n\sqrt{2}| < 2\,, \quad \text{(b)}\quad |m - n\sqrt{2}| < \sqrt{2}\gamma\,, \\ &\text{(c)}\quad |p - q\sqrt{2}| < 2\alpha\,, \quad \text{(d)}\quad |p - q\sqrt{2}| < \sqrt{2}\beta\,. \end{aligned} \tag{10}$$

The length types (a,c) are connected with the matrices ρ_γ of the type $\rho^{(1)}$ whereas (b,d) are connected with $\rho^{(2)}$. Considering the connection between the type of ρ_γ and the code of γ we see that the type of the orbit length is uniquely defined by parities of the number of bounces λ_γ and $\mu_\gamma + \nu_\gamma$, see Table I.

TABLE I: Orbit length types for different parities of symbol numbers

	λ even	λ odd
$(\mu+\nu)$ even	a	c
$(\mu+\nu)$ odd	d	b

It is elementary to prove that the numbers in r.h.s. of the equations (a),(b) in (8) cannot be equal unless they are zero; the same is true for (c),(d) in (9). Therefore a length multiplet can never simultaneously contain orbits of the types (a) and (b) or (c) and (d). On the other hand, an orbit with an even number of symbols n_γ can have the same length as one with odd n_γ, and such equality happens for some rare combinations of $u_{k,R}, v_{k,R}$; these are the cases of *gu*-degeneracy mentioned above. Equating one of l_a, l_b to one of l_c, l_d we obtain an equation for u, v equivalent to two diophantine equations for the integers m, n, p, q. (Not all solutions of these equations correspond to really existing length multiplets since (8), (9) are only necessary conditions.) A careful analysis of the latter equations (see below for details) for the cases $l_a = l_c$ or $l_b = l_d$ reveals that the solutions form an equidistant sequence $l_k^{(1)} = s^I k, \quad k = 1, 2, \ldots$ with $s^I = 2\,\mathrm{arsinh}\, 2^{-1/4}$. Similarly solutions for $l_a = l_d$ or $l_b = l_c$ are described by $l_k^{(2)} = s^{II} k, \quad k = 1, 2, \ldots,$ with $s^{II} = 2\,\mathrm{arsinh}\sqrt{1+\sqrt{2}}$. E.g., the exceptional length $l = 10.6999964 = 7s^I$ pertains to two multiplets, (b) with $m = 138, n = 97$, and (d) with $p = 88, q = 63$.

Since the exceptional lengths appear in equidistant sequences the multiplets with *gu*-degeneracy are exponentially outnumbered, as the length l grows, by the multiplets of definite type and hence parity of λ and $\mu+\nu$. The *gu*-degenerate multiplets can thus be discarded for lengths corresponding to a finite fraction of T_H.

To conclude, we have shown that "all" (all save for a negligible fraction of multiplets) orbits with the same length of the $T^*(2,3,8)$ billiard have the same parity of the number of bounces against the longer leg L of the triangle; this is also true for the parity of the sum of the number of bounces against the sides M, N. As a result "all" orbits with the same length in the arithmetical case have the same Maslov index such that all diagonal terms in the form factor are positive. A similar mechanism must exist in all other arithmetical systems with non-vanishing Maslov phases. In pseudo-arithmetical systems the Maslov phases of degenerate orbits are uncorrelated.

Off-diagonal corrections: We based our reasoning on the diagonal approximation. But the essence of our argument carries over to the higher-order terms of the τ-expansion valid for $\tau < 1$ [4] as well as for the behavior at times exceeding the Heisenberg time [5], as far as the pseudo-arithmetical case is concerned: The high multiplicity of length multiplets of orbits is rendered irrelevant by destructive interference of random Maslov phases.

For arithmetical systems the interplay of high length degeneracy and bunches of orbits that very nearly coincide in configuration space, apart from reconnections in close self-encounters, may be more difficult to capture. On the other hand, the Hecke symmetries intuitively suggest (near) Poissonian level statistics.

Technical note on exceptional multiplets: We briefly indicate how existence and uniqueness of the exceptional lengths can be ascertained, for the four possible cases $l_a = l_c, l_a = l_d, l_b = l_c, l_b = l_d$ which we shall refer to as ac, ad, bc, bd. Starting with the case ac we infer from Eqs. (8,9) that the exceptional lengths obey

$$e^{\frac{l}{2}} = \frac{u+\alpha v}{2}, \quad e^{-\frac{l}{2}} = \frac{u-\alpha v}{2}\,. \tag{11}$$

Now take two solutions $u_0 + \alpha v_0$ and $u + \alpha v$ corresponding to the lengths l_0, l. Then since $\exp\left[\pm\left(\frac{l_0}{2}+\frac{l}{2}\right)\right] = \exp\left(\pm\frac{l_0}{2}\right)\exp\left(\pm\frac{l}{2}\right)$, the product

$$u' + \alpha v' = \frac{u_0+\alpha v_0}{2}\frac{u+\alpha v}{2} \tag{12}$$

is a solution with length $l' = l + l_0$, provided the integers in the rhs are divisible by 2.

The numerically found solution of smallest length $l_0^I = 2s^I = 3.05714183896200$ was $u_0 = 2+2\sqrt{2}, v_0 = 4+2\sqrt{2}$, or $m_0 = 2, n_0 = 2, p_0 = 4, q_0 = 2$. Substituting the latter into (12) we obtain

$$\begin{aligned} m' &= m + 2n + 2q, \\ n' &= m + n + p, \\ p' &= 2m + 2n + p + 2q, \\ q' &= m + 2n + p + q; \end{aligned} \tag{13}$$

obviously if m, n, p, q are even then so are the primed numbers. We thus face the sequence of solutions $\left(\frac{u_0+\alpha v_0}{2}\right)^k$ with the equidistant lengths kl_0^I.

The transformation (13) can be inverted; the doubly primed numbers

$$\begin{aligned} m'' &= m + 2n - 2q, \\ n'' &= m + n - p, \\ p'' &= -2m - 2n + p + 2q, \\ q'' &= -m - 2n + p + q; \end{aligned} \tag{14}$$

yield an orbit of length $l'' = l - l_0^I$ such that a ladder of decreasing lengths is obtained.

In order to make sure that the transformations (13,14) really yield orbits we must check that the inequalities (a,c) in (10) are not violated. To that end we note that the variables $x \equiv (m - n\sqrt{2})/2$, $y \equiv (p - q\sqrt{2})/2\alpha$ span an invariant subspace of the transformations (13,14); the ensuing transformation $(x, y) \to (x', y')$ is a two dimensional rotation preserving $x^2 + y^2$. Since the aforementioned minimal-length solution $m_0 = 2, n_0 = 2, p_0 =$

$4, q_0 = 2$ has the property $x^2+y^2 = 1$ and since that "normalization" is preserved we indeed conclude the preservation of the inequalities under study.

Finally, let us demonstrate that there are no lengths of the type ac outside the equidistant-length ladder just established. Momentarily assuming the existence of such a freak length we can apply the length reducing transformation (14) until arriving at a reduced length $l'' \in [0, l_0^I]$ which must obey

$$1 \ < \ \cosh\frac{l''}{2} = \frac{m''+n''\sqrt{2}}{2} < 1+\sqrt{2} = \cosh\frac{l_0^I}{2}\,, \qquad (15)$$
$$0 \ < \ \sinh\frac{l''}{2} = \left(p''+\sqrt{2}q''\right)\frac{\alpha}{2} < \left(2+\sqrt{2}\right)\alpha = \sinh\frac{l_0^I}{2}\,.$$

On the other hand, the conservation of x^2+y^2 subjects the freak to $(x'')^2+(y'')^2 = \left(m''-n''\sqrt{2}\right)^2/4 + \left(p''-q''\sqrt{2}\right)^2/4\alpha^2 = x^2+y^2 \le 2$, and thus to $\left|m''-n''\sqrt{2}\right| < 2\sqrt{2}$, $\left|p''-q''\sqrt{2}\right| < 2\sqrt{2}\alpha$. Upon checking the remaining finite number of quadruplets of integers m'', n'', p'', q'' with lengths inside the interval $0 < l'' < l_0^I$ we conclude that no freak can exist.

The discussion of case ad proceeds in close analogy with the previously treated ac. The only differences are the replacements of (i) the quartic irrationality α by β and (ii) the minimal length l_0^I by $l_0^{II} = 2s^{II} = 4.8969$, the latter corresponding to $u_0 = 6+4\sqrt{2}, \quad v_0 = 4+4\sqrt{2}$. All other lengths are given by $kl_0^{II}, \quad k = 1, 2, \ldots$. The proof repeats the previous case word for word.

To treat case bd we employ $\gamma = \alpha\beta$ to write the equations for the lengths as

$$e^{\frac{l}{2}} = \frac{\beta\left(\alpha u+v\right)}{2}, \quad e^{-\frac{l}{2}} = \frac{\beta\left(\alpha u-v\right)}{2}\,. \qquad (16)$$

The smallest length corresponds to $u_m = 2+\sqrt{2}, v_m = \sqrt{2}$ and is exactly s^I, i.e., half of the smallest length of case ac. Solutions of (16) do not have in general the group property of the case ac; in particular, if l is an exceptional length of the type bd then $2l$ is not. Indeed, squaring the rhs of (16) we get $\sqrt{2}\left(\alpha u \pm v\right)^2/4$ which cannot be an algebraic number of the type $\beta\left(\alpha u+v\right)/2$. On the other hand, all odd multiples of l do belong to the admissible type. The same reasoning as in the cases ac, ad shows that all solutions can be represented as $(k+1/2)\, l_0^I$, $k = 0, 1, \ldots$.

The remaining case bc is related to bd by the replacements $\alpha \leftrightarrow \beta$ and $l_0^I \to l_0^{II}$. All solutions can be written as $(k+1/2)\, l_0^{II}$, $k = 0, 1, \ldots$.

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